

# Acyclic edge coloring of graphs

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## Abstract

An *acyclic edge coloring* of a graph  $G$  is a proper edge coloring such that the subgraph induced by any two color classes is a linear forest (an acyclic graph with maximum degree two); in other words, there are no bichromatic cycles in  $G$ . The *acyclic chromatic index*  $\chi'_a(G)$  of a graph  $G$  is the least number of colors needed in any acyclic edge coloring of  $G$ . Fiamčík conjectured that  $\chi'_a(G) \leq \Delta(G) + 2$ , where  $\Delta(G)$  is the maximum degree of  $G$ . A  $\kappa$ -*deletion-minimal* graph is one with maximum degree at most  $\kappa$ , and it has no acyclic edge coloring with  $\kappa$  colors, but every proper subgraph admits an acyclic edge coloring with  $\kappa$  colors. In this paper, we give some structure lemmas on the  $\kappa$ -deletion-minimal graphs. As applications, we prove that every planar  $G$  with maximum average degree less than four admits an acyclic edge coloring with  $\Delta(G) + 2$  colors; let  $G$  be a planar graph without triangles adjacent to cycles of length 3 and 4, and every 5-cycle has at most three edges contained in triangles, then  $G$  admits an acyclic edge coloring with  $\Delta(G) + 2$  colors. As immediate consequences, we obtain some known results as corollaries. Hopefully, the structure lemmas will be useful on the acyclic edge coloring problem.

## 1 Introduction

All graphs considered are finite, simple and undirected. An *acyclic edge coloring* of a graph  $G$  is a proper edge coloring such that the subgraph induced by any two color classes is a linear forest (an acyclic graph with maximum degree two); in other words, there are no bichromatic cycles in  $G$ . The *acyclic chromatic index*  $\chi'_a(G)$  of a graph  $G$  is the least number of colors needed in any acyclic edge coloring of  $G$ . We denote the minimum and maximum degrees of vertices of  $G$  by  $\delta(G)$  and  $\Delta(G)$ , respectively.

Fiamčík [11] stated the following conjecture in 1978, which is well-known as Acyclic Edge Coloring Conjecture, and Alon et al. [2] restated it in 2001.

**Conjecture 1.** For every graph  $G$ , we have  $\chi'_a(G) \leq \Delta(G) + 2$ .

Alon, McDiarmid and Reed [1] proved that the acyclic chromatic index of a graph  $G$  is at most  $64\Delta(G)$ . Molloy and Reed [19] improved the upper bound on the acyclic chromatic index to  $16\Delta(G)$ . Ndreca et al. [21] improved the upper bound to  $\lceil 9.62(\Delta(G) - 1) \rceil$ ; recently, Esperet and Parreau [10] further improved it to  $4\Delta(G)$  by using the so-called entropy compression method. Note that  $\chi'_a(G) \leq 3$  if  $\Delta(G) = 2$ . Burštejn [9] proved that every graph with maximum degree 4 has an acyclic vertex coloring

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with 5 colors. Since an acyclic edge coloring of a graph  $G$  is in fact an acyclic vertex coloring of its line graph  $L(G)$ , and the maximum degree of a line graph  $L(G)$  of a subcubic graph  $G$  is at most 4, it follows that  $\chi'_a(G) \leq 5$  if  $\Delta(G) = 3$ . Hence, [Conjecture 1](#) holds for  $\Delta(G) \leq 3$ . Furthermore, Andersen et al. [3] proved that the acyclic chromatic index of a connected subcubic graph  $G$  is 4, unless  $G$  is  $K_4$  or  $K_{3,3}$ ; the acyclic chromatic index of  $K_4$  and  $K_{3,3}$  is 5. This conjecture has also been verified for some special classes of graphs. Muthu et al. [20] proved that  $\chi'_a(G) \leq \Delta(G) + 1$  for outerplanar graphs; Hou et al. [18] proved that  $\chi'_a(G) = \Delta(G)$  for outerplanar graphs. The conjecture is also true for the planar graphs with girth at least five [8, 17]; for the planar graphs with girth at least four [22].

Fiedorowicz et al. [13] proved that  $\chi'_a(G) \leq 2\Delta(G) + 29$  for every planar graphs  $G$ . Basavaraju et al. [6] showed that the acyclic chromatic index of a planar graph  $G$  is at most  $\Delta(G) + 12$ . Furthermore, Guan et al. [14] improved it to  $\Delta(G) + 10$ ; and Wang et al. [25] make it to  $\Delta(G) + 7$ .

The *maximum average degree*  $\text{mad}(G)$  of a graph  $G$  is the largest average degree of its subgraph, that is,

$$\text{mad}(G) = \max_{H \subseteq G} \left\{ \frac{2|E(H)|}{|V(H)|} \right\}.$$

For the graphs with small maximum average degree, we have known the following result.

**Theorem 1.1** ([13]). If  $G$  is a graph with  $\text{mad}(G) < 4$ , then  $\chi'_a(G) \leq \Delta(G) + 6$ .

**Theorem 1.2** ([4]). If  $G$  is a graph with  $\text{mad}(G) < 4$ , then  $\chi'_a(G) \leq \Delta(G) + 3$ .

Recently, Wang et al. [26] published the following result, but the proof has some rather serious gaps.

**Theorem 1.3.** If  $G$  is a graph with  $\text{mad}(G) < 4$ , then  $\chi'_a(G) \leq \Delta(G) + 2$ .

A  $\kappa$ -*deletion-minimal* graph is one with maximum degree at most  $\kappa$ , and it has no acyclic edge coloring with  $\kappa$  colors, but every proper subgraph admits an acyclic edge coloring with  $\kappa$  colors. A graph property  $\mathcal{P}$  is *deletion-closed* if  $\mathcal{P}$  is closed under removal of edges.

In section 3, we give some structure lemmas on  $\kappa$ -deletion-minimal graphs. In section 4, we present some results as applications of these structure lemmas, one of which is a correct proof of [Theorem 1.3](#).

## 2 Preliminary

Let  $G$  be a graph and  $H$  a subgraph of  $G$ . An acyclic edge coloring of  $H$  is an *partial acyclic edge coloring* of  $G$ . Denote  $\mathcal{U}_\phi(v)$  the colors which appears at  $v$  with respect to  $\phi$ , and let  $C_\phi(v) = [\kappa] \setminus \mathcal{U}_\phi(v)$ . Let  $S_\phi(uv) = \mathcal{U}_\phi(v) \setminus \{\phi(uv)\}$ . An  $(\alpha, \beta)$ -maximal bichromatic path with respect to  $\phi$  is a maximal path whose edges are colored by  $\alpha$  and  $\beta$  alternately. In other words, an  $(\alpha, \beta)$ -maximal bichromatic path with respect to  $\phi$  is a component of the graph induced by the edges with color  $\alpha$  or  $\beta$ . An  $(\alpha, \beta, u, v)$ -critical path with respect to  $\phi$  is an  $(\alpha, \beta)$ -maximal bichromatic path with respect to  $\phi$  which starts at  $u$  with color  $\alpha$  and ends at  $v$ .

Let  $\phi$  be a partial acyclic edge coloring of  $G$ . A color  $\alpha$  is *candidate* for an edge  $e$  in  $G$  with respect to a partial edge coloring of  $G$  if none of the adjacent edges of  $e$  is colored with  $\alpha$ . A candidate color  $\alpha$  is *valid* for an edge  $e$  if assigning the color  $\alpha$  to  $e$  does not result any bichromatic cycle in  $G$ .

**Fact 1.** Given partial acyclic edge coloring of  $G$  and two colors  $\alpha, \beta$ , there exists at most one  $(\alpha, \beta)$ -maximal path containing a particular vertex  $v$ .

For an acyclic edge coloring  $\phi$  of  $G - uv$ , let  $W(uv) = \{u_i \mid uu_i \in E(G) \text{ and } \phi(uu_i) \in \mathcal{U}_\phi(v)\}$ .

**Fact 2.** Let  $G$  be a  $\kappa$ -deletion-minimal graph and  $uv$  an edge of  $G$ . If  $\phi$  is an acyclic edge coloring of  $G - uv$ , then every candidate color for  $uv$  is not valid. Furthermore, if  $\mathcal{U}_\phi(u) \cap \mathcal{U}_\phi(v) = \emptyset$ , then  $\deg(u) + \deg(v) = \kappa + 2$ ; if  $|\mathcal{U}_\phi(u) \cap \mathcal{U}_\phi(v)| = s$ , then  $\deg(u) + \deg(v) + \sum_{w \in W(uv)} \deg(w) \geq \kappa + 2s + 2$ .

The *degree* of a vertex  $v$  in  $G$ , denoted by  $\deg(v)$ , is the number of incident edges of  $G$ . A vertex of degree  $k$  is called a  $k$ -vertex, and a vertex of degree at most or at least  $k$  is called a  $k^-$ - or  $k^+$ -vertex, respectively. Let  $[\kappa]$  stands for the set  $\{1, \dots, \kappa\}$ .

### 3 Structure Lemmas

**Lemma 1.** If  $G$  is a  $\kappa$ -deletion-minimal graph, then  $G$  is 2-connected.

**Lemma 2** ([16]). If  $G$  is a  $\kappa$ -deletion-minimal graph and  $v_0$  be a vertex of  $G$ , then

$$\sum_{w \in N_G(v_0)} \deg(w) \geq \kappa + \deg(v_0).$$

**Lemma 3.** Let  $G$  be a  $\kappa$ -deletion-minimal graph. If  $v$  is adjacent to a 2-vertex  $v_0$  and  $N_G(v_0) = \{v, w\}$ , then  $v$  is adjacent to at least  $\kappa - \deg(w) + 1$  vertices of degree at least  $\kappa - \deg(v) + 2$ . Moreover,

- (i) if  $v$  and  $w$  are adjacent, then  $v$  is adjacent to at least  $\kappa - \deg(w) + 2$  vertices of degree at least  $\kappa - \deg(v) + 2$ , and  $\deg(v) \geq \kappa - \deg(w) + 3$ ;
- (ii) if  $\kappa \geq \Delta(G) + 2$  and  $v$  is adjacent to precisely  $\kappa - \Delta(G) + 1$  vertices of degree at least  $\kappa - \Delta(G) + 2$ , then  $v$  is adjacent to at most  $\deg(v) + \Delta(G) - \kappa - 3$  vertices of degree two and  $\deg(v) \geq \kappa - \Delta(G) + 4$ .

**Proof.** Since  $G$  is  $\kappa$ -deletion-minimal, it follows that  $G - v_0$  admits an acyclic edge coloring  $\phi$  with  $\kappa$  colors. Without loss of generality, assume that  $N_G(v_0) = \{v_0, v_1, \dots, v_n\}$  and  $\phi(vv_i) = i$  for  $i = 1, 2, \dots, n$ . If  $\deg(v) = \kappa$ , then there is nothing to prove by Lemma 2. So we may assume  $\deg(v) < \kappa$ . Assume that  $C_\phi(w) \cap C_\phi(v) \neq \emptyset$ . Choose colors  $\theta_1 \in C_\phi(w) \cap C_\phi(v)$  and  $\theta_2 \in C_\phi(v) \setminus \{\theta_1\}$ , and extend  $\phi$  by assigning  $\theta_1$  to  $wv_0$  and assigning  $\theta_2$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. So we may assume that  $C_\phi(w) \cap C_\phi(v) = \emptyset$ . Consequently,  $C_\phi(w) \subseteq \mathcal{U}_\phi(v)$  and  $C_\phi(v) \subseteq \mathcal{U}_\phi(w)$ . So we may assume that  $C_\phi(w) = \{1, \dots, m\}$ , where  $m = \kappa - \deg(w) + 1$ .

Hence, for every  $i$  in  $C_\phi(w)$  and every  $j$  in  $C_\phi(v)$ , there exists an  $(i, j, v, w)$ -critical path with respect to  $\phi$ ; otherwise, extend  $\phi$  by assigning  $i$  to  $wv_0$  and assigning  $j$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. Thus, for every  $i \in \{1, \dots, m\}$ , we have  $\{n+1, \dots, \kappa\} \subseteq \mathcal{U}(v_i)$ . Note that  $|C_\phi(w)| = \kappa - \deg(w) + 1$  and  $|C_\phi(v)| = \kappa - \deg(v) + 1$ . Hence, the vertex  $v$  is adjacent to at least  $\kappa - \deg(w) + 1$  vertices of degree at least  $\kappa - \deg(v) + 2$ .

(i) By Lemma 2, we have  $\deg(w) \geq \kappa + 2 - \deg(v)$ , and thus  $w$  is a  $(\kappa - \deg(v) + 2)$ -vertex. Note that  $w \notin \{v_1, v_2, \dots, v_m\}$ , so (i) holds.

(ii) From now on, we assume that  $\kappa \geq \Delta(G) + 2$  and  $v$  is adjacent to precisely  $\kappa - \Delta(G) + 1$  vertices of degree at least  $\kappa - \Delta(G) + 2$ . It follows that  $\deg(w) = \Delta(G)$ . Hence, the precisely  $\kappa - \Delta(G) + 1$  vertices of degree at least  $\kappa - \Delta(G) + 2$  are  $v_1, \dots, v_m$ . By contradiction, we may assume that  $v_i$  is a 2-vertex and  $N_G(v_i) = \{v, w_i\}$  for  $i = m+2, \dots, n$ . Note that  $\{v_0, v_{m+2}, \dots, v_n\}$  is an independent set in  $G$  by the 2-connectedness, thus  $w \notin \{v_{m+2}, \dots, v_n\}$ .

**Claim 1.** For every  $i$  in  $\{1, \dots, m\}$  and every  $j$  in  $\{m+1, \dots, n\}$ , there exists an  $(i, j, v, w)$ -critical path with respect to  $\phi$ .

**Proof.** Suppose that there exists an  $i$  in  $\{1, \dots, m\}$  and a  $j$  in  $\{m+1, \dots, n\}$  such that there is no  $(i, j, v, w)$ -critical path with respect to  $\phi$ . Modify  $\phi$  by erasing the color  $j$  from  $vv_j$  and assigning  $i$  to  $wv_0$  and assigning  $j$  to  $vv_0$ , we obtain an acyclic edge coloring  $\psi$  of  $G - vv_j$  with  $\kappa$  colors. By Fact 2,  $\mathcal{U}_\psi(v) \cap \mathcal{U}_\psi(v_j) \neq \emptyset$  since  $\deg(v) + \deg(v_j) \leq \deg(v) + (\kappa - \Delta(G) + 1) < \kappa + 2$ .

Suppose that  $S_\psi(vv_j) \cap \{m+1, \dots, n\} = \emptyset$ . Extend  $\psi$  by assigning a color in  $C_\phi(v) \setminus S_\phi(vv_j)$  to  $vv_j$ , we obtain an acyclic edge coloring of  $G$ , a contradiction. Hence, we may assume that  $S_\psi(vv_j) \cap \{m+1, \dots, n\} \neq \emptyset$ .

0. If  $\deg(v_j) = 2$ , then  $|\mathcal{U}_\psi(v) \cap \mathcal{U}_\psi(v_j)| = 1$ , but  $\deg(v) + \deg(v_j) + \sum_{x \in W(vv_j)} \deg(x) \leq \deg(v) + 2 + (\kappa - \Delta(G) + 1) < \kappa + 4$ , which contradicts Fact 2. Hence,  $v_j$  must be a  $3^+$ -vertex and  $j = m + 1$ . Without loss of generality, we may assume that  $n \in S_\psi(vv_{m+1})$ . By Fact 2, we have  $S_\psi(vv_{m+1}) \cap \{1, \dots, m\} \neq \emptyset$ . Hence,  $\{n + 1, \dots, \kappa\} \setminus (S_\psi(vv_{m+1}) \cup \{\phi(v_i w_i) \mid v_i \in W(vv_j) \text{ and } i \geq m + 2\}) \neq \emptyset$ ; but every color in this set is valid for  $vv_{m+1}$  with respect to  $\psi$ , a contradiction.  $\square$

Hence,  $\deg(v_i) = \Delta(G)$  and  $S_\phi(vv_i) = \{m + 1, \dots, \kappa\}$  for every  $i$  in  $\{1, \dots, m\}$ . Let  $\phi_{i,j}$  be the proper edge coloring obtained from  $\phi$  by exchanging the colors on  $vv_i$  and  $vv_j$  for  $1 \leq i < j \leq m$ .

**Claim 2.** There exists  $i_0$  and  $j_0$  in  $\{1, \dots, m\}$  such that the coloring  $\phi_{i_0, j_0}$  has no bichromatic cycle containing the edge  $vv_{m+1}$ .

**Proof.** The vertex  $v_{m+1}$  is a vertex of degree at most  $m$ , there are at most  $m - 1$  critical paths with respect to  $\phi$  containing  $vv_{m+1}$  and passing through  $v_{m+1}$ . Hence, there exists a vertex in  $\{v_1, \dots, v_m\}$ , say  $v_1$ , such that there is no  $(m + 1, i, v, v_1)$ -critical path with respect to  $\phi$  for every color  $i$  in  $[\kappa]$ . Note that  $m \geq 3$ ; therefore,  $\phi_{1,2}$  is the desired coloring if there is no  $(m + 1, 1, v, v_2)$ -critical path with respect to  $\phi$ ; otherwise,  $\phi_{1,3}$  is the desired coloring.  $\square$

Without loss of generality, we may assume that  $\phi_{1,2}$  is the coloring obtained in Claim 2. If  $\phi_{1,2}$  is an acyclic edge coloring of  $G - v_0$ , then extend  $\phi_{1,2}$  by assigning  $\kappa$  to  $vv_0$  and assigning 1 to  $wv_0$ , we obtain an acyclic edge coloring of  $G$ , which is a contradiction. Hence, the coloring  $\phi_{1,2}$  is not an acyclic edge coloring of  $G - v_0$ , and then it admits bichromatic cycle containing  $vv_1$  or  $vv_2$  but not containing  $vv_{m+1}$ . Let  $T_1 = \{\theta \mid \theta \in \{m + 2, \dots, \kappa\} \text{ and there exists an } (1, \theta)\text{-bichromatic cycle containing } vv_2 \text{ with respect to } \phi_{1,2}\}$ . Let  $T_2 = \{\theta \mid \theta \in \{m + 2, \dots, \kappa\} \text{ and there exists an } (2, \theta)\text{-bichromatic cycle containing } vv_1 \text{ with respect to } \phi_{1,2}\}$ . Consequently, we have  $T_1 \cup T_2 \neq \emptyset$ . Note that  $T_1 \cap T_2 = \emptyset$ , since all the vertices  $v_i$  with  $i \geq m + 2$  are 2-vertices.

**Claim 3.** We may assume that  $|T_1| \leq 1$  and  $|T_2| \leq 1$ .

**Proof.** Without loss of generality, we may assume that  $T_1 = \{m + 2, \dots, r\}$  and  $T_2 = \{s, \dots, n\}$ , where  $r < s$  (note that  $T_1 \cap T_2 = \emptyset$ ). If  $|T_1| > 1$ , then recolor  $vv_i$  with  $i + 1$  for  $m + 2 \leq i \leq r - 1$  and recolor  $vv_r$  with  $m + 2$ . Similarly, if  $|T_2| > 1$ , then recolor  $vv_j$  with  $j + 1$  for  $s \leq j \leq n - 1$  and recolor  $vv_n$  with  $s$ . Finally, we obtain a proper edge coloring  $\psi'$  of  $G - v_0$  satisfying  $|T_1| \leq 1$  and  $|T_2| \leq 1$  with respect to  $\psi'$ , but it has no bichromatic cycle containing  $vv_{m+1}$ .  $\square$

By symmetry, we may assume that  $|T_1| \geq |T_2|$  and  $T_1 = \{m + 2\}$ . The following proof is divided into two cases:

**Case 1.**  $|T_2| = 0$ .

Modify  $\phi_{1,2}$  by assigning 1 to  $wv_0$  and assigning  $m + 2$  to  $vv_0$ , and erasing the colors on  $vv_{m+2}$  and  $v_{m+2}w_{m+2}$ , we obtain an acyclic edge coloring  $\psi^*$  of  $G - v_{m+2}$ . By similar arguments as above, we know that  $\deg(w_{m+2}) = \Delta(G)$  and  $C_{\psi^*}(w_{m+2}) = \{1, \dots, m\}$ . Extend  $\psi^*$  by reassigning  $\kappa$  to  $vv_{m+2}$  and reassigning  $m$  to  $v_{m+2}w_{m+2}$  (note that  $w_{m+2} \neq w$  since  $1 \notin \mathcal{U}_\phi(w)$  and  $1 \in \mathcal{U}_\phi(w_{m+2})$ ), we obtain an acyclic edge coloring of  $G$ , a contradiction.

**Case 2.**  $|T_2| = 1$ .

Without loss of generality, we may assume that  $T_2 = \{m + 3\}$ . Obviously,  $\phi(v_{m+2}w_{m+2}) = 1$  and  $\phi(v_{m+3}w_{m+3}) = 2$ . Note that  $w \neq w_{m+2}$  and  $w \neq w_{m+3}$  since  $C_\phi(w) = \{1, \dots, m\}$  but  $1 \in \mathcal{U}_\phi(w_{m+2})$  and  $2 \in \mathcal{U}_\phi(w_{m+3})$ . Suppose that there is a color  $\theta$  in  $C_\phi(w_{m+3})$  with  $\theta \in \{3, \dots, m\} \cup \{n + 1, \dots, \kappa\}$ . Modify  $\phi_{1,2}$  by reassigning  $\theta$  to  $v_{m+3}w_{m+3}$ , the resulting coloring has similar properties as  $\phi_{1,2}$  and then we go back to Case 1. Hence,  $C_\phi(w_{m+3}) \subseteq \{1\} \cup \{m + 1, \dots, n\}$ , so we may assume that there exists a color  $\theta$  in  $C_\phi(w_{m+3})$  with  $m + 1 \leq \theta \leq n$ . Since  $\deg(v) + \deg(v_{m+3}) + \deg(v_\theta) \leq \Delta(G) + 2 + (\kappa - \Delta(G) + 1) < \kappa + 4$ , we can

modify  $\phi_{1,2}$  by reassigning  $\theta$  to  $v_{m+3}w_{m+3}$  and reassigning a suitable color to  $vv_{m+3}$ , such that the resulting proper edge coloring has no bichromatic cycle containing  $vv_{m+3}$ , then we go back to Case 1.  $\square$

**Lemma 4.** Let  $G$  be a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G) + 2$ . If  $v_0$  is a 2-vertex, then every neighbor of  $v_0$  has degree at least  $\kappa - \Delta(G) + 4$ .

**Proof.** Let  $N_G(v_0) = \{v, w\}$ . By contradiction and Lemma 3, suppose that  $v$  is a  $(\kappa - \Delta(G) + 3)$ -vertex and  $N_G(v) = \{v_0, v_1, \dots, v_n\}$ , where  $n = \kappa - \Delta(G) + 2$ . Since  $G$  is  $\kappa$ -deletion-minimal, it follows that  $G - v_0$  admits an acyclic edge coloring  $\phi$  with  $\kappa$  colors. Without loss of generality, assume that  $\phi(vv_i) = i$  for  $i = 1, \dots, n$ , and then  $C_\phi(v) = \{n+1, \dots, \kappa\}$ . Assume that  $C_\phi(w) \cap C_\phi(v) \neq \emptyset$ . Choose colors  $\theta_1 \in C_\phi(w) \cap C_\phi(v)$  and  $\theta_2 \in C_\phi(w) \setminus \{\theta_1\}$ , and extend  $\phi$  by assigning  $\theta_1$  to  $vv_0$  and assigning  $\theta_2$  to  $wv_0$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. So we may assume that  $C_\phi(w) \cap C_\phi(v) = \emptyset$ . Consequently,  $C_\phi(w) \subseteq \{1, \dots, n\}$  and  $\{n+1, \dots, \kappa\} \subseteq \mathcal{U}_\phi(w)$ . Hence, for every  $i$  in  $C_\phi(w)$  and each  $j$  in  $C_\phi(v)$ , there exists an  $(i, j, v, w)$ -critical path with respect to  $\phi$ ; otherwise, extend  $\phi$  by assigning  $i$  to  $vv_0$  and assigning  $j$  to  $wv_0$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. Thus, for every  $i \in C_\phi(w)$ , we have  $\{n+1, \dots, \kappa\} \subseteq \mathcal{U}_\phi(v_i)$ . Without loss of generality, we may assume that  $\{1, \dots, n-1\} \subseteq C_\phi(w) \subseteq \{1, \dots, n\}$ .

For every pair of  $i$  and  $j$  with  $1 \leq i < j \leq n$ , modify  $\phi$  by exchanging the colors on  $vv_i$  and  $vv_j$ , we obtain an edge coloring of  $G - v_0$ , we denote this coloring by  $\phi_{i,j}$ . Note that  $1 \leq i \leq n-1$  and  $i \in C_\phi(w)$ .

**Claim 1.** If the coloring  $\phi_{i,j}$  is a proper edge coloring of  $G - v_0$ , then it must contain bichromatic cycles.

**Proof.** If  $\phi_{i,j}$  is an acyclic edge coloring of  $G - v_0$ , then extend  $\phi_{i,j}$  by assigning  $i$  to  $wv_0$  and assigning  $\kappa$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction.  $\square$

**Claim 2.**  $\{n+1, \dots, \kappa\} \subseteq \mathcal{U}_\phi(v_n)$ .

**Proof.** By contradiction, assume that  $j_0 \in \{n+1, \dots, \kappa\} \cap C_\phi(v_n)$ . Note that  $w \neq v_n$  since  $j_0 \in \mathcal{U}_\phi(w)$  but  $j_0 \notin \mathcal{U}_\phi(v_n)$ . Modify  $\phi$  by assigning 1 to  $wv_0$  and assigning  $n$  to  $vv_0$  and reassigning  $j_0$  to  $vv_n$ , we obtain a proper edge coloring of  $G$ , then it must contain an  $(n, 1)$ -bichromatic cycle containing  $vv_0$ ; otherwise, it is an acyclic edge coloring of  $G$ , a contradiction. Hence,  $n \in \mathcal{U}_\phi(v_1)$  and  $S_\phi(vv_1) = \{n, \dots, \kappa\}$ . Similarly, we can prove that  $S_\phi(vv_i) = \{n, \dots, \kappa\}$  for  $i = 1, \dots, n-1$ . Hence, all the colorings  $\phi_{i,j}$  with  $1 \leq i < j \leq n-1$  are proper edge coloring of  $G - v_0$ , and then there exists bichromatic cycles containing  $vv_n$  with respect to each of these colorings by Claim 1. Without loss of generality, we may assume that there exists an  $(n, 2)$ -bichromatic cycle with respect to  $\phi_{1,2}$ ; in other words, there exists an  $(n, 2, v, v_1)$ -critical path with respect to  $\phi$ . Hence, for every  $t$  with  $2 \leq t \leq n-1$ , there is no  $(n, 2, v, v_t)$ -critical path with respect to  $\phi$ . Suppose that  $n \geq 5$ . Hence, there exists an  $(n, 3)$ -bichromatic cycle containing  $vv_n$  with respect to  $\phi_{2,3}$  and there exists an  $(n, 4)$ -bichromatic cycle containing  $vv_n$  with respect to  $\phi_{2,4}$ ; in other words, there exists an  $(n, 3, v, v_2)$ - and  $(n, 4, v, v_2)$ -critical path with respect to  $\phi$ . But  $\phi_{3,4}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts Claim 1. Thus,  $n = 4$ . There exists an  $(4, 3)$ -bichromatic cycle containing  $vv_4$  with respect to  $\phi_{2,3}$ , and then there exists an  $(4, 3, v, v_2)$ -critical path with respect to  $\phi$ . Hence, there is no  $(4, 3)$ -bichromatic cycle containing  $vv_4$  with respect to  $\phi_{1,3}$ , and thus there is an  $(4, 1, v, v_3)$ -critical path with respect to  $\phi$ . Modify  $\phi$  by reassigning 3, 1, 2 to  $vv_1, vv_2, vv_3$  respectively, and assigning 1 to  $wv_0$  and assigning  $\kappa$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction.  $\square$

Now, we have  $\{n+1, \dots, \kappa\} \subseteq S_\phi(vv_i)$  for  $i = 1, \dots, n$ , it follows that  $|S_\phi(vv_i) \cap \{1, \dots, n\}| \leq 1$  for  $i = 1, \dots, n$ .

**Claim 3.** If  $S_\phi(vv_n) \cap \{1, \dots, n\} = \emptyset$ , then  $S_\phi(vv_i) = \{n, \dots, \kappa\}$  for  $i = 1, \dots, n-1$ .

**Proof.** By contradiction and the symmetry, assume that  $n \notin S_\phi(vv_{n-1})$ . By Claim 1, the proper edge coloring  $\phi_{n-1,n}$  must contain bichromatic cycles. Note that there is no bichromatic cycle containing  $vv_n$ , it follows that there exists an  $(n, i)$ -bichromatic cycle containing  $vv_{n-1}$  with respect to  $\phi_{n-1,n}$ , where  $i \leq n-2$ .

By symmetry, assume that there exists an  $(n, 1)$ -bichromatic cycle containing  $vv_{n-1}$  with respect to  $\phi_{n-1,n}$ . Hence,  $S_\phi(vv_{n-1}) \cap \{1, \dots, n\} = \{1\}$  and  $S_\phi(vv_1) \cap \{1, \dots, n\} = \{n\}$ . If there exists a vertex  $v_j$  with  $2 \leq j \leq n-2$  such that  $n \in S_\phi(vv_j)$ , then  $\phi_{1,j}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#). Hence, for every vertex  $v_j$  with  $2 \leq j \leq n-2$ , we have  $n \notin S_\phi(vv_j)$ . Since there exists an  $(1, n, v, v_{n-1})$ -critical path with respect to  $\phi$ , thus  $\phi_{2,n}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#).  $\square$

By [Claim 3](#), if  $S_\phi(vv_n) \cap \{1, \dots, n\} = \emptyset$ , then  $S_\phi(vv_i) = \{n, \dots, \kappa\}$  for  $i = 1, \dots, n-1$ , and then  $\phi_{1,2}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#). So we may assume that  $S_\phi(vv_n) \cap \{1, \dots, n\} \neq \emptyset$ . By symmetry, assume that  $n-1 \in S_\phi(vv_n)$ .

**Case 1.** There are at least two vertices  $v_i$  with  $i \in \{1, \dots, n-2\}$  such that  $n \in S_\phi(vv_i)$ .

Without loss of generality, we may assume that  $n \in \mathcal{U}_\phi(v_1) \cap \mathcal{U}_\phi(v_2)$ . The coloring  $\phi_{1,2}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#).

**Case 2.** For every vertex  $v_i$  with  $1 \leq i \leq n-2$ , we have  $n \in C_\phi(v_i)$ .

If  $1 \in \mathcal{U}_\phi(v_{n-1})$ , then the proper edge coloring  $\phi_{2,n}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#). So we may assume that  $1 \notin \mathcal{U}_\phi(v_{n-1})$ . By [Claim 1](#), the proper edge coloring  $\phi_{1,n}$  must contain  $(n, n-1)$ -bichromatic cycle containing  $vv_1$  and  $n \in S_\phi(vv_{n-1})$ , but  $\phi_{2,n}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#).

**Case 3.** There is only one vertex  $v_i$  with  $1 \leq i \leq n-2$  such that  $n \in S_\phi(vv_i)$ . By symmetry, we may assume that  $n \in \mathcal{U}_\phi(v_2)$ .

**Subcase 3.1.**  $\{1, n\} \subseteq C_\phi(v_{n-1})$ .

By [Claim 1](#), the proper edge coloring  $\phi_{1,n}$  must contain  $(n, 2)$ -bichromatic cycle containing  $vv_1$ . Hence,  $2 \in S_\phi(vv_1)$ . If  $n = 4$ , then  $\phi_{1,3}$  is an acyclic edge coloring of  $G - v_0$ , which contradicts [Claim 1](#). So we may assume that  $n \geq 5$ . The proper edge coloring  $\phi_{1,n-1}$  must contain a bichromatic cycle containing  $vv_{n-1}$ , say  $(1, i)$ -bichromatic cycle, where  $i \in \{3, \dots, n-2\}$ . Without loss of generality, we may assume that there is  $(1, 3)$ -bichromatic cycle with respect to  $\phi_{1,n-1}$ . Hence,  $1 \in S_\phi(vv_3)$  and  $3 \in S_\phi(vv_{n-1})$ . Modify  $\phi$  by reassigning 3, 1, 2 to  $vv_1, vv_2, vv_3$  respectively, and assigning 2 to  $wv_0$  and assigning  $\kappa$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction.

**Subcase 3.2.**  $\{1, n\} \cap S_\phi(vv_{n-1}) \neq \emptyset$ .

Suppose that  $2 \in \mathcal{U}_\phi(v_1)$ . Modify  $\phi$  by reassigning  $n-1, 1, 2$  to  $vv_1, vv_2, vv_{n-1}$  respectively, we obtain an acyclic edge coloring of  $G - v_0$ , and extend it by assigning 2 to  $wv_0$  and assigning  $\kappa$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction. So we may assume that  $2 \notin \mathcal{U}_\phi(v_1)$ . By [Claim 1](#), the proper edge coloring  $\phi_{1,2}$  admits a bichromatic cycle containing  $vv_1$ . Thus,  $n \geq 5$ . So we may assume that there is a  $(2, 3)$ -bichromatic cycle with respect to  $\phi_{1,2}$ , hence  $3 \in S_\phi(vv_1)$  and  $2 \in S_\phi(vv_3)$ . Modify  $\phi$  by reassigning 2, 3, 1 to  $vv_1, vv_2, vv_3$  respectively, and assigning 2 to  $wv_0$  and assigning  $\kappa$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction.  $\square$

**Lemma 5.** Let  $G$  be a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G)+1$ . If  $v_0$  is a 2-vertex, then every neighbor of  $v_0$  has degree at least four.

**Proof.** Let  $N_G(v_0) = \{v, w\}$ . By contradiction and [Lemma 3](#), suppose that  $v$  is a 3-vertex and  $N_G(v) = \{v_0, v_1, v_2\}$ . Since  $G$  is  $\kappa$ -deletion-minimal, it follows that  $G - v_0$  admits an acyclic edge coloring  $\phi$  with  $\kappa$  colors. Without loss of generality, assume that  $\phi(vv_i) = i$  for  $i = 1, 2$ , and then  $C_\phi(v) = \{3, \dots, \kappa\}$ . Assume that  $C_\phi(w) \cap \{3, \dots, \kappa\} \neq \emptyset$ . Choose colors  $\theta_1 \in C_\phi(w) \cap \{3, \dots, \kappa\}$  and  $\theta_2 \in C_\phi(w) \setminus \{\theta_1\}$ , and extend  $\phi$  by assigning  $\theta_1$  to  $vv_0$  and assigning  $\theta_2$  to  $wv_0$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. So we may assume that  $C_\phi(w) \cap \{3, \dots, \kappa\} = \emptyset$ . Consequently,  $C_\phi(w) = \{1, 2\}$  and  $\deg(w) = \Delta(G)$ . Hence, for  $i = 1, 2$  and each  $j$  in  $\{3, \dots, \kappa\}$ , there exists an  $(i, j, v, w)$ -critical path

with respect to  $\phi$ ; otherwise, extend  $\phi$  by assigning  $i$  to  $wv_0$  and assigning  $j$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. Thus, for every  $i = 1, 2$ , we have  $\deg(v_i) = \Delta(G)$  and  $S_\phi(vv_i) = \{3, \dots, \kappa\}$ . Modify  $\phi$  by exchanging the colors on  $vv_1$  and  $vv_2$ , we obtain a new acyclic edge coloring of  $G - v_0$ ; extend this coloring by assigning 1 to  $wv_0$  and  $\kappa$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction.  $\square$

**Lemma 6.** Let  $G$  be a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G) + 2$ , and let  $v$  be a 3-vertex with  $N(v) = \{w, v_1, v_2\}$ . If  $\deg(w) = \kappa - \Delta(G) + 2$ , then  $G$  has the following properties:

- (a) for any acyclic edge coloring  $\phi$  of  $G - vw$  with  $\kappa$  colors, we have  $|\mathcal{U}_\phi(v) \cap \mathcal{U}_\phi(w)| = 1$ . Without loss of generality, assume that  $\mathcal{U}_\phi(v) \cap \mathcal{U}_\phi(w) = \{\phi(vv_1)\}$ .
- (b)  $\deg(v_1) = \Delta(G) \geq \deg(v_2) \geq \kappa - \Delta(G) + 3$ ;
- (c) the edge  $wv$  is not contained in any triangle in  $G$  and  $w$  is adjacent to exactly one  $3^-$ -vertex, say  $v$ ;
- (d) the vertex  $v_1$  is adjacent to at least  $\kappa - \deg(v_2) + 1$  vertices of degree at least  $\kappa - \Delta(G) + 2$ ;
- (e) the vertex  $v_2$  is adjacent to at least  $\kappa - \Delta(G)$  vertices of degree at least  $\kappa - \deg(v_2) + 2$ ;
- (f) the vertex  $v_2$  is adjacent to at least  $\kappa - \Delta(G) + 1$  vertices of degree at least four.

**Proof.** Let  $n = \kappa - \Delta(G) + 1$ . Note that  $n \geq 3$ . Without loss of generality, assume that  $N(w) = \{v, x_1, \dots, x_n\}$  and  $\phi(wx_i) = i$  for  $i = 1, \dots, n$ . If  $\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v) = \emptyset$ , then  $\deg(w) + \deg(v) \leq \Delta(G) + 3 < \kappa + 2$ , which contradicts Fact 2. Hence,  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| \geq 1$ . Without loss of generality, assume that  $\phi(vv_1) = 1$ .

**Claim 1.**  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| = 1$ .

**Proof.** By contradiction, without loss of generality, we may assume that  $\phi(vv_2) = 2$ . For any  $i \in \{n + 1, \dots, \kappa\}$ , there exists an  $(1, i, v, w)$ -critical path with respect to  $\phi$  or there exists an  $(2, i, v, w)$ -critical path with respect to  $\phi$ .

Let  $T_1 = \{i \mid i \in \{n + 1, \dots, \kappa\} \text{ and there exists an } (1, i, v, w)\text{-critical path with respect to } \phi\}$

and  $T_2 = \{j \mid j \in \{n + 1, \dots, \kappa\} \text{ and there exists an } (2, j, v, w)\text{-critical path with respect to } \phi\}$ .

Hence,  $T_1 \cup T_2 = \{n + 1, \dots, \kappa\}$ .

**Case 1.** Either  $S_\phi(vv_1) \not\supseteq \{n + 1, \dots, \kappa\}$  or  $S_\phi(vv_2) \not\supseteq \{n + 1, \dots, \kappa\}$ .

Without loss of generality, assume that  $n + 1 \notin S_\phi(vv_2)$ . By the assumption, it follows that there exists an  $(1, n + 1, v, w)$ -critical path with respect to  $\phi$ . Modify  $\phi$  by reassigning  $n + 1$  to  $vv_2$ , we obtain a new acyclic edge coloring  $\psi$  of  $G - vw$  with  $|\mathcal{U}_\psi(w) \cap \mathcal{U}_\psi(v)| = 1$ . Note that every candidate color for  $vw$  is not valid; in other words, for every  $\theta \in \{n + 2, \dots, \kappa\}$ , there exists an  $(1, \theta, v, w)$ -critical path with respect to  $\psi$  (the same with respect to  $\phi$ ). Consequently,  $S_\phi(vv_1) = S_\phi(wx_1) = \{n + 1, \dots, \kappa\}$ . Modify  $\phi$  by reassigning 3 to  $vv_1$ , we obtain another acyclic edge coloring  $\phi^*$  of  $G - vw$ . Similarly, we can prove that, for every  $\theta \in \{n + 1, \dots, \kappa\}$ , there exists an  $(3, \theta, v, w)$ -critical path with respect to  $\phi^*$ , and  $S_\phi(vv_1) = S_\phi(wx_3) = \{n + 1, \dots, \kappa\}$ . Modify  $\phi$  by exchanging the colors on  $wx_1$  and  $wx_3$ , we obtain a new acyclic edge coloring of  $G - vw$ , but now  $\kappa$  is valid for  $vw$ , a contradiction.

**Case 2.**  $S_\phi(vv_1) \supseteq \{n + 1, \dots, \kappa\}$  and  $S_\phi(vv_2) \supseteq \{n + 1, \dots, \kappa\}$ .



In fact,  $S_\phi(vv_1) = \{n+1, \dots, \kappa\}$  and  $S_\phi(vv_2) = \{n+1, \dots, \kappa\}$ . By symmetry, we may assume that  $T_1 \neq \emptyset$ . Modify  $\phi$  by exchanging the colors on  $vv_1$  and  $vv_2$ , we obtain a new acyclic edge coloring of  $G - vw$ . Note that every candidate color for  $vw$  is not valid. In other words, if assign an arbitrary color  $\theta_2$  in  $T_2$  to  $vw$ , then there exists an  $(1, \theta_2)$ -bichromatic cycle containing  $vw$ ; if assign an arbitrary color  $\theta_1$  in  $T_1$  to  $vw$ , then there exists an  $(2, \theta_1)$ -bichromatic cycle containing  $vw$ . Now, we have  $S_\phi(wx_1) = S_\phi(wx_2) = T_1 \cup T_2 = \{n+1, \dots, \kappa\}$ . Modify  $\phi$  by reassigning 3 to  $vv_1$  and reassigning 1 to  $vv_2$ , and choosing an arbitrary color  $\theta_1$  in  $T_1$  and assigning it to  $vw$ , the resulting proper edge coloring has an  $(3, \theta_1)$ -bichromatic cycle containing  $vw$ . Modify  $\phi$  by exchanging the colors on  $wx_1$  and  $wx_2$ , and reassigning 3 to  $vv_2$ , we obtain an acyclic edge coloring of  $G - vw$ . But every color in  $T_1$  is valid for  $vw$  with respect to this acyclic edge coloring of  $G - vw$ , which derive a contradiction.  $\square$

Therefore, for any acyclic edge coloring  $\phi$  of  $G - vw$  with  $\kappa$  colors, we have  $|\mathcal{U}_\phi(v) \cap \mathcal{U}_\phi(w)| = 1$ .

Without loss of generality, let  $\phi(vv_2) = n+1$ . For every color  $\theta$  in  $\{n+2, \dots, \kappa\}$  for  $vw$ , there exists an  $(1, \theta, v, w)$ -critical path with respect to  $\phi$ ; otherwise, the color  $\theta$  is valid for  $vw$ , a contradiction. Hence,  $\mathcal{U}_\phi(v_1) \supseteq \{1, n+2, \dots, \kappa\}$  and  $\mathcal{U}_\phi(x_1) \supseteq \{1, n+2, \dots, \kappa\}$ . Consequently, there is no  $(1, \theta, v, v_2)$ -critical path with respect to  $\phi$  for any  $\theta \in \{n+2, \dots, \kappa\}$ .

**Claim 2.** There is an  $(1, n+1, v, w)$ -critical path (not including  $v_2$ ) with respect to  $\phi$ .

**Proof.** Suppose that there is no  $(1, n+1, v, w)$ -critical path with respect to  $\phi$ . Erase  $n+1$  from  $vv_2$  and assign  $n+1$  to  $vw$ , it yields an acyclic edge coloring  $\psi^*$  of  $G - vv_2$ . Note that  $\mathcal{U}_{\psi^*}(v) \cap \mathcal{U}_{\psi^*}(v_2) = \{1\}$  by Fact 2. If there is a color  $\theta$  in  $\{n+2, \dots, \kappa\} \cap C_{\psi^*}(v_2)$ , then  $\theta$  is valid for  $vv_2$  with respect to  $\psi^*$  since there is no  $(1, \theta, v, v_2)$ -critical path with respect to  $\phi$ , a contradiction. It follows that  $\mathcal{U}_\phi(v_2) \supseteq \{1, n+1, \dots, \kappa\}$ ; in fact, we have  $\mathcal{U}_\phi(v_2) = \{1, n+1, \dots, \kappa\}$  and  $C_\phi(v_2) = \{2, \dots, n\}$ . If  $C_\phi(v_1) \cap \{2, \dots, n\} \neq \emptyset$ , then choose an  $\alpha \in C_\phi(v_1) \cap \{2, \dots, n\}$  and assign  $\alpha$  to  $vv_2$ , it yields an acyclic edge coloring of  $G$  with  $\kappa$  colors, which is a contradiction. Therefore,  $\{2, \dots, n\} \subseteq \mathcal{U}_\phi(v_1)$  and  $\mathcal{U}_\phi(v_1) \supseteq \{1, \dots, n, n+2, \dots, \kappa\}$ , and then  $\deg(v_1) \geq \kappa - 1 \geq \Delta(G) + 1$ , which derive a contradiction.  $\square$

Now, for every  $\theta$  in  $\{n+1, \dots, \kappa\}$ , there exists an  $(1, \theta, v, w)$ -critical path with respect to  $\phi$ . Hence,  $\mathcal{U}_\phi(v_1) \supseteq \{1, n+1, \dots, \kappa\}$  and  $\mathcal{U}_\phi(x_1) \supseteq \{1, n+1, \dots, \kappa\}$ . Furthermore,  $\deg(v_1) = \deg(x_1) = \Delta(G)$  and  $\mathcal{U}_\phi(v_1) = \mathcal{U}_\phi(x_1) = \{1, n+1, \dots, \kappa\}$ . Note that  $wv_1 \notin E(G)$  since  $S_\phi(vw) \cap S_\phi(vv_1) = \emptyset$ .

**Claim 3.**  $\{1, \dots, n\} \cap C_\phi(v_2) = \emptyset$ .

**Proof.** Suppose, towards a contradiction, that there is a color  $\alpha$  in  $\{1, \dots, n\} \cap C_\phi(v_2)$ . Modify  $\phi$  by reassigning  $\alpha$  to  $vv_2$  and reassigning a color  $\beta$  in  $\{1, \dots, n\} \setminus \{\alpha\}$  to  $vv_1$ , we obtain an acyclic edge coloring  $\phi^*$  of  $G - vw$  with  $\kappa$  colors. But  $|\mathcal{U}_{\phi^*}(w) \cap \mathcal{U}_{\phi^*}(v)| = 2$ , which contradicts Lemma 6 (a).  $\square$

Consequently,  $\mathcal{U}_\phi(v_2) \supseteq \{1, \dots, n+1\}$  and  $C_\phi(v_2) \subseteq \{n+2, \dots, \kappa\}$ . Without loss of generality, we may assume that  $C_\phi(v_2) = \{n+2, \dots, m\}$ . Note that  $m \geq 2(\kappa - \Delta(G) + 1)$  since  $|C_\phi(v_2)| \geq \kappa - \Delta(G)$ .

**Claim 4.** For every  $\theta \in \{2, \dots, n\}$  and each  $\lambda \in \{n+1, \dots, m\}$ , if reassigning  $\lambda$  to  $vv_2$ , then there exists an  $(\lambda, \theta, v, v_1)$ -critical path (not including  $w$ ) with respect to  $\phi$ .

**Proof.** Suppose that there exists a color  $\theta^* \in \{2, \dots, n\}$  and a  $\lambda^* \in \{n+1, \dots, m\}$  such that there is no  $(\lambda^*, \theta^*, v, v_1)$ -critical path if reassigning  $\lambda^*$  to  $vv_2$ . Modify  $\phi$  by reassigning  $\theta^*$  to  $vv_1$  and reassigning  $\lambda^*$  to  $vv_2$ , we obtain an acyclic edge coloring of  $G - vw$  with  $\kappa$  colors; by similar arguments as in Claim 1, we have  $\mathcal{U}_\phi(x_{\theta^*}) = \{\theta^*, n+1, \dots, \kappa\}$ . Therefore, modify  $\phi$  by exchanging the colors on  $wx_1$  and  $wx_{\theta^*}$ , we obtain an acyclic edge coloring of  $G - vw$ , but  $\kappa$  is valid for  $vw$  with respect to this coloring, which derive a contradiction. Note that  $w$  does not on the  $(\lambda, \theta, v, v_1)$ -critical path if reassign  $\lambda$  to  $vv_2$ , for otherwise  $\lambda \in \mathcal{U}_\phi(w)$ .  $\square$



Without loss of generality, let  $N_G(v_1) = \{v, y_{n+1}, y_{n+2}, \dots, y_\kappa\}$  and let  $\phi(v_1 y_\lambda) = \lambda$  for every  $\lambda = n+1, \dots, \kappa$ . Hence, for every  $\lambda \in \{n+1, \dots, m\}$ , we have  $\mathcal{U}_\phi(y_\lambda) \supseteq \{1, \dots, n, \lambda\}$  and  $\deg(y_\lambda) \geq n+1$ , and then  $v_1$  is adjacent to at least  $\kappa - \deg(v_2) + 1$  neighbors of degree at least  $n+1$ . Meanwhile, for every  $\theta \in \{2, \dots, n\}$ , we have  $\mathcal{U}_\phi(z_\theta) \supseteq \{n+1, \dots, m, \theta\}$  and  $\deg(z_\theta) \geq n+1$ , and then there are at least  $\kappa - \Delta(G)$  neighbors of  $v_2$  having degree at least  $\kappa - \deg(v_2) + 2$ . Note that  $wv_2 \notin E(G)$ ; otherwise, if  $\phi(wv_2) = 1$ , then it contradicts [Claim 1](#); if  $\phi(wv_2) \in \{2, \dots, n\}$ , then it contradicts [Claim 4](#). Hence, the edge  $wv$  is not contained in any triangle.

**Claim 5.**  $\deg(v_2) \geq \kappa - \Delta(G) + 3$ .

**Proof.** If  $\deg(v_2) = \kappa - \Delta(G) + 2 = n+1$ , then  $\mathcal{U}_\phi(v_2) = \{1, \dots, n+1\}$  and  $m = \kappa$ . By the above arguments, we have  $S_\phi(v_2 z_i) = \{n+1, \dots, \kappa\}$  for every vertex  $z_i$  with  $2 \leq i \leq n$ . Modify  $\phi$  by exchanging the colors on  $v_2 z_2$  and  $v_2 z_3$ , reassigning 2 to  $vv_1$ , we obtain an acyclic edge coloring of  $G - vw$ , but  $\kappa$  is valid for  $vw$  with respect to this coloring, a contradiction.  $\square$

Consequently,  $\deg(v_1) = \Delta(G) \geq \deg(v_2) \geq \kappa - \Delta(G) + 3$ .

**Claim 6.** The vertices in  $\{x_2, \dots, x_n\}$  are all  $4^+$ -vertices.

**Proof.** By contradiction and the symmetry, suppose that  $x_2$  is a  $3^-$ -vertex. By [Lemma 3](#), the vertex  $x_2$  is a 3-vertex. Modify  $\phi$  by erasing the color on  $wx_2$  and assigning 2 to  $vw$ , we obtain an acyclic edge coloring  $\phi^*$  of  $G - wx_2$ . By [Lemma 6](#), we have  $|\mathcal{U}_{\phi^*}(w) \cap \mathcal{U}_{\phi^*}(x_2)| = 1$ . If  $\mathcal{U}_{\phi^*}(w) \cap \mathcal{U}_{\phi^*}(x_2) = \{1\}$ , then there is a color  $\theta$  with  $\theta \geq n+1$  such that it is valid for  $wx_2$ , a contradiction. If  $\mathcal{U}_{\phi^*}(w) \cap \mathcal{U}_{\phi^*}(x_2) = \{3\} \subseteq \{3, \dots, n\}$ , then we can similarly prove that  $\mathcal{U}_{\phi^*}(x_3) = \{3, n+1, \dots, \kappa\}$ . Modify  $\phi$  by exchanging colors on  $wx_1$  and  $wx_3$ , we obtain a new acyclic edge coloring of  $G - vw$ , but  $\kappa$  is valid for  $vw$  with respect to this coloring, a contradiction. Therefore, the vertices in  $\{x_2, \dots, x_n\}$  are all  $4^+$ -vertices.  $\square$

In what follows, suppose that  $v_2$  is only adjacent to precisely  $\kappa - \Delta$  vertices of degree at least four, say  $z_2, \dots, z_n$ . By [Lemma 3](#), all the other neighbors of  $v_2$  has degree three. Modify  $\phi$  by reassigning 2 to  $vv_1$ , reassigning 1 to  $vv_2$  and assigning  $n+1$  to  $vw$ , erasing the color 1 from  $v_2 z_1$ , we obtain an acyclic edge coloring  $\pi$  of  $G - v_2 z_1$ . By Fact 2, we have  $\mathcal{U}_\pi(v_2) \cap \mathcal{U}_\pi(z_1) \neq \emptyset$ . Let  $N_G(z_1) = \{z'_1, z''_1, v_2\}$ . If  $S_\pi(v_2 z_1) \subseteq \{2, \dots, m\}$ , then every color in  $\{n+1, \dots, m\} \setminus S_\pi(v_2 z_1)$  is valid for  $v_2 z_1$  with respect to  $\pi$ , a contradiction. So we may assume that  $\phi(z_1 z'_1) = m+1$ .

Suppose that  $\phi(z_1 z'_1) \in \{m+2, \dots, \kappa\}$ , without loss of generality, assume that  $\phi(z_1 z'_1) = m+2$ . Hence, for every color  $\theta$  in  $\{n+1, \dots, m\}$ , there exists an  $(m+1, \theta, z_1, v_2)$ -critical path or there exists an  $(m+2, \theta, z_1, v_2)$ -critical path with respect to  $\pi$ . Without loss of generality, we may assume that there exists an  $(m+1, n+1, z_1, v_2)$ -critical path and an  $(m+1, n+2, z_1, v_2)$ -critical path with respect to  $\pi$ . Hence,  $S_\pi(v_2 z_{m+1}) = \{n+1, n+2\}$ . Modify  $\pi$  by reassigning  $n+3$  to  $v_2 z_{m+1}$ , we obtain a new acyclic edge coloring  $\pi'$  of  $G - v_2 z_1$ . The candidate color  $n+1$  and  $n+2$  for  $v_2 z_1$  is not valid with respect to  $\pi'$ , it follows that  $S_{\pi'}(v_2 z_{m+2}) = \{n+1, n+2\}$ . Now, we have  $S_\pi(v_2 z_{m+1}) = S_{\pi'}(v_2 z_{m+2}) = \{n+1, n+2\}$ , but  $n+3$  is valid for  $v_2 z_1$  with respect to  $\pi$ , a contradiction. So we may assume that  $\phi(z_1 z'_1) \in \{2, \dots, m\}$ . Since all the candidate colors for  $v_2 z_1$  is not valid with respect to  $\pi$ , it follows that, for every color  $\theta$  in  $\{n+1, \dots, m\} \setminus \{\phi(z_1 z'_1)\}$ , there exists an  $(m+1, \theta, z_1, v_2)$ -critical path with respect to  $\pi$ ; consequently, we have  $m = 6, n = 3, \deg(v_2) = \Delta(G)$  and  $S_\phi(v_2 z_7) \cup \{\phi(z_1 z'_1)\} = \{4, 5, 6\}$ .

Without loss of generality, assume that  $\phi(z_1 z'_1) = 6$  and  $\phi(z_7 z'_7) = 4$  and  $\phi(z_7 z''_7) = 5$ . There exists an  $(2, 7, v, v_2)$ -critical path with respect to  $\pi$ ; otherwise, modify  $\pi$  by reassigning 7 to  $vv_2$  and reassigning 6 to  $v_2 z_7$  and reassigning 1 to  $v_2 z_1$ , we obtain an acyclic edge coloring of  $G$ , a contradiction. Similarly, there exists an  $(3, 7, v, v_2)$ -critical path with respect to  $\pi$  if reassigning 3 to  $vv_1$ . Hence,  $S_\phi(v_2 z_2) \cap S_\phi(v_2 z_3) \supseteq \{4, 5, 6, 7\}$ . If  $\Delta(G) = 5$ , then  $S_\phi(v_2 z_2) = S_\phi(v_2 z_3) = \{4, 5, 6, 7\}$ , modify  $\phi$  by exchanging the colors on  $v_2 z_2$  and  $v_2 z_3$ , reassigning 2 to  $vv_1$  and reassigning 4 to  $vv_2$  and reassigning 5 to  $wv$ , we obtain an acyclic edge coloring of  $G$ , a contradiction. So we may assume that  $\Delta(G) \geq 6$  and  $\kappa = \Delta(G) + 2 \geq 8$ .

If there exists a vertex  $z_i$  with  $i \geq 8$  such that  $S_\phi(v_2z_i) \subseteq \{1, \dots, 6\}$ , then modify  $\pi$  by exchanging the colors on  $v_2z_7$  and  $v_2z_i$ , we obtain an acyclic edge coloring of  $G - v_2z_1$ , but now 4 is valid for  $v_2z_1$ , a contradiction. So we may assume that  $|S_\phi(v_2z_i) \cap \{1, \dots, 6\}| \leq 1$  for every  $i$  in  $\{8, \dots, \kappa\}$ .

Suppose that  $7 \in S_\phi(v_2z_i)$  for some  $i$  with  $i \geq 8$ , without loss of generality, assume that  $\phi(z_iz_i'') = 7$ . Modify  $\pi$  by erasing  $i$  from  $v_2z_i$  and reassigning  $i$  to  $v_2z_1$ , we obtain an acyclic edge coloring  $\pi_2$  of  $G - v_2z_i$ . For every  $\theta$  in  $\{4, 5, 6\} \setminus \{\phi(z_iz_i')\}$ , there is no  $(7, \theta, z_i, v_2)$ -critical path with respect to  $\pi_2$ , hence there exists an  $(\phi(z_iz_i'), \theta, z_i, v_2)$ -critical path. Hence,  $\phi(z_iz_i') \geq 8$ . Consequently, there exists an  $t$  with  $t \geq 8$  and  $\phi(v_2z_t) = \phi(z_iz_i') = t$  such that  $S_\phi(v_2z_t) \supseteq \{4, 5, 6\} \setminus \{\phi(z_iz_i')\} = \{4, 5, 6\}$ , which contradicts the fact that  $v_t$  is a 3-vertex. Therefore,  $7 \notin S_\phi(v_2z_i)$  for every  $i \geq 8$ . Modify  $\pi$  by exchanging the colors on  $v_2z_7$  and  $v_2z_8$ , we obtain an acyclic edge coloring of  $G - v_2z_1$ , but now 4 is valid for  $v_2z_1$ , a contradiction.  $\square$

**Lemma 7** (Hou et al. [16]). Let  $G$  be a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G) + 2$ . If  $v$  is a 3-vertex, then every neighbor of  $v$  is a  $(\kappa - \Delta(G) + 2)^+$ -vertex.

A vertex is a *special* 3-vertex if it is a 3-vertex and it is adjacent to some  $(\kappa - \Delta(G) + 2)$ -vertices; otherwise, it is called *normal* 3-vertex. In other words, a vertex is a normal 3-vertex if it is a 3-vertex and every neighbor of  $v$  is a  $(\kappa - \Delta(G) + 3)^+$ -vertex by Lemma 7.

**Lemma 8.** If  $G$  is a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G) + 2$  and  $w$  is a  $(\kappa - \Delta(G) + 3)$ -vertex, then  $w$  is adjacent to at most  $\kappa - \Delta(G) + 1$  vertices of degree three.

**Proof.** Let  $\tau = \kappa - \Delta(G) + 3$ . To derive a contradiction, assume that  $w$  is adjacent to at least  $\tau - 1$  vertices of degree three. Let  $N(w) = \{v, x_1, \dots, x_{\tau-1}\}$  and  $v$  be a 3-vertex and  $N(v) = \{w, v_1, v_2\}$ . Without loss of generality, let  $\phi$  be an acyclic edge coloring of  $G - vw$  with  $\phi(wx_i) = i$  for  $i = 1, \dots, \tau - 1$ . If  $\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v) = \emptyset$ , then  $\deg(w) + \deg(v) \leq \Delta(G) + 3 < \kappa + 2$ , which contradicts Fact 2. Hence,  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| \geq 1$ . Without loss of generality, assume that  $\phi(vv_1) = 1$ .

**Case 1.**  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| = 1$ .

Without loss of generality, let  $\phi(vv_2) = \tau$ . For every color  $\theta$  in  $\{\tau + 1, \dots, \kappa\}$ , there exists an  $(1, \theta, v, w)$ -critical path with respect to  $\phi$ ; otherwise, we can assign some  $\theta$  to  $vw$  and obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction. Hence,  $\mathcal{U}_\phi(v_1) \supseteq \{1, \tau + 1, \dots, \kappa\}$  and  $\mathcal{U}_\phi(x_1) \supseteq \{1, \tau + 1, \dots, \kappa\}$ . Consequently, there exists no  $(1, \theta, v, v_2)$ -critical path with respect to  $\phi$  for any  $\theta \in \{\tau + 1, \dots, \kappa\}$ .

**Claim 1.** There is an  $(1, \tau, v, w)$ -critical path (not including  $v_2$ ) with respect to  $\phi$ .

**Proof.** Suppose that there is no  $(1, \tau, v, w)$ -critical path with respect to  $\phi$ . Modify  $\phi$  by erasing  $\tau$  from  $vv_2$  and assigning  $\tau$  to  $vw$ , it yields an acyclic edge coloring  $\phi^*$  of  $G - vv_2$ . Note that  $\mathcal{U}_{\phi^*}(v) \cap \mathcal{U}_{\phi^*}(v_2) = \{1\}$ . If there is a color  $\theta$  in  $\{\tau + 1, \dots, \kappa\}$  which is missed at  $v_2$ , then  $\theta$  is valid for  $vv_2$  with respect to  $\phi^*$  since there is no  $(1, \theta, v, v_2)$ -critical path with respect to  $\phi$ , a contradiction. It follows that  $\mathcal{U}_{\phi^*}(v_2) \supseteq \{1, \tau, \dots, \kappa\}$  and then  $C_{\phi^*}(v_2) \subseteq \{2, \dots, \tau - 1\}$ . If  $C_{\phi^*}(v_1) \cap C_{\phi^*}(v_2) \neq \emptyset$ , then choose a color  $\alpha \in C_{\phi^*}(v_1) \cap C_{\phi^*}(v_2)$  and extend  $\phi^*$  by assigning  $\alpha$  to  $vv_2$ , it yields an acyclic edge coloring of  $G$  with  $\kappa$  colors, which is a contradiction. Hence  $C_{\phi^*}(v_2) \subseteq \mathcal{U}_{\phi^*}(v_1)$  and  $C_{\phi^*}(v_1) \subseteq \mathcal{U}_{\phi^*}(v_2)$ . Note that  $C_{\phi^*}(v_2) \subseteq \{2, \dots, \tau - 1\}$  and  $|C_{\phi^*}(v_2)| \geq \kappa - \Delta(G)$ , it follows that  $\deg(w) + \deg(v_1) = \kappa + |\mathcal{U}_\phi(v_1) \cap \{1, \dots, \tau\}| \geq \kappa + 1 + (\kappa - \Delta(G)) = \kappa - 2 + \tau \geq \Delta(G) + \tau$ . Hence,  $\deg(v_1) = \deg(v_2) = \Delta(G)$  and  $\kappa = \Delta(G) + 2$ , and then  $\tau = 5$  and  $|C_{\phi^*}(v_2)| = 2$ . Without loss of generality, assume that  $C_{\phi^*}(v_2) = \{2, 3\}$ , and then  $\mathcal{U}_{\phi^*}(v_1) = \{1, 2, 3, 6, \dots, \kappa\}$  and  $\mathcal{U}_{\phi^*}(v_2) = \{1, 4, 5, 6, \dots, \kappa\}$ . Modify  $\phi$  by reassigning 4 to  $vv_1$ , we obtain an acyclic edge coloring  $\phi_{4,5}$  of  $G - vw$ ; hence, for every color  $\theta$  in  $\{6, \dots, \kappa\}$ , there exists an  $(4, \theta, v, w)$ -critical path with respect to  $\phi_{4,5}$ , and thus  $\mathcal{U}_{\phi}(x_4) \supseteq \{4, 6, \dots, \kappa\}$ . Modify  $\phi$  by reassigning 5 to  $vv_1$  and reassigning 2 to  $vv_2$ , we obtain an acyclic edge coloring  $\phi_{5,2}$  of  $G - vw$ ; hence, for every color  $\theta$  in  $\{6, \dots, \kappa\}$ , there exists an  $(2, \theta, v, w)$ -critical path with respect to  $\phi_{5,2}$ , and thus  $\mathcal{U}_{\phi}(x_2) \supseteq \{2, 6, \dots, \kappa\}$ . Modify  $\phi$  by reassigning 5 to  $vv_1$  and reassigning 3 to  $vv_2$ , we obtain an acyclic edge coloring  $\phi_{5,3}$  of  $G - vw$ ; hence, for every color  $\theta$  in  $\{6, \dots, \kappa\}$ , there exists an  $(3, \theta, v, w)$ -critical path with respect to  $\phi_{5,3}$ , and thus  $\mathcal{U}_{\phi}(x_3) \supseteq \{3, 6, \dots, \kappa\}$ . That is,  $S_\phi(wx_i) \supseteq \{6, \dots, \kappa\}$  for

$i = 1, \dots, 4$ . Recall that  $w$  is adjacent to at least four 3-vertices, including  $x_i$  and  $x_j$ . Hence,  $\kappa = 7$  and  $S_\phi(w x_i) = S_\phi(w x_j) = \{6, 7\}$ . Modify  $\phi_{5,i}$  by exchanging colors on  $w x_i$  and  $w x_j$ , and assigning 6 to  $vw$ , we obtain an acyclic edge coloring of  $G$  with  $\kappa$  colors, a contradiction.  $\square$

Hence,  $\mathcal{U}_\phi(v_1) \cap \mathcal{U}_\phi(x_1) \supseteq \{1, \tau, \dots, \kappa\}$ . The degree of  $x_1$  is at least four since  $|\{1, \tau, \dots, \kappa\}| \geq 4$ , it follows that  $x_2, \dots, x_{\tau-1}$  are all 3-vertices.

Note that  $C_\phi(v_1) \subseteq \{2, \dots, \tau-1\}$ . Without loss of generality, assume that  $\{2, \dots, \tau-2\} \subseteq C_\phi(v_1) \subseteq \{2, \dots, \tau-1\}$ . Modify  $\phi$  by erasing the color on  $w x_2$  and assigning 2 to  $vw$ , we obtain an acyclic edge coloring  $\psi$  of  $G - w x_2$ . If  $\mathcal{U}_\psi(x_2) \cap \mathcal{U}_\psi(w) = \{i\}$ , then  $x_i$  is a  $4^+$ -vertex by the above arguments, and then  $i = 1$ . Extend  $\psi$  by assigning  $\theta$  to  $w x_2$ , where  $\theta$  is in  $\{\tau, \dots, \kappa\} \cap C_\phi(x_2)$ , we obtain an acyclic edge coloring of  $G$ , which is a contradiction. Hence,  $|\mathcal{U}_\psi(x_2) \cap \mathcal{U}_\psi(w)| = 2$ ; in other words,  $|\mathcal{U}_\phi(x_2) \cap \{1, \dots, \tau-1\}| = 3$ . Similarly, we can prove that  $|\mathcal{U}_\phi(x_i) \cap \{1, \dots, \tau-1\}| = 3$  for  $i = 2, \dots, \tau-1$ .

If  $1 \notin \mathcal{U}_\phi(x_2)$ , then every color in  $\{\tau, \dots, \kappa\}$  is valid for  $w x_2$  with respect to  $\psi$ , a contradiction. So we may assume that  $1 \in \mathcal{U}_\phi(x_2)$ , and let  $\mathcal{U}_\phi(x_2) = \{1, 2, r\}$ . For every  $i \in \{\tau, \dots, \kappa\}$ , there exists no  $(1, i, w, x_2)$ -critical path with respect to  $\psi$ , then there exists an  $(r, i, w, x_2)$ -critical path with respect to  $\psi$ , we obtain  $\deg(x_r) \geq 4$ , which is a contradiction.

**Case 2.**  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| = 2$ .

Without loss of generality, we may assume that  $\phi(vv_2) = 2$ . For any  $i \in \{\tau, \dots, \kappa\}$ , there exists an  $(1, i, v, w)$ -critical path with respect to  $\phi$  or there exists an  $(2, i, v, w)$ -critical path with respect to  $\phi$ .

**Subcase 2.1.** Either  $S_\phi(vv_1) \not\supseteq \{\tau, \dots, \kappa\}$  or  $S_\phi(vv_2) \not\supseteq \{\tau, \dots, \kappa\}$ .

By symmetry, assume that  $\tau \notin S_\phi(vv_2)$ . Note that  $\tau$  is not valid for  $vw$  with respect to  $\phi$ , it follows that there exists an  $(1, \tau, v, w)$ -critical path with respect to  $\phi$ , and then there is no  $(1, \tau, v, v_2)$ -critical path with respect to  $\phi$ . Modify  $\phi$  by reassigning  $\tau$  to  $vv_2$ , we obtain an acyclic edge coloring  $\phi'$  of  $G - vw$  with  $|\mathcal{U}_{\phi'}(w) \cap \mathcal{U}_{\phi'}(v)| = 1$  and then we go back to Case 1.

**Subcase 2.2.**  $S_\phi(vv_1) \supseteq \{\tau, \dots, \kappa\}$  and  $S_\phi(vv_2) \supseteq \{\tau, \dots, \kappa\}$ .

Suppose that  $x_1$  and  $x_2$  are all 3-vertices. For every color  $i$  in  $\{\tau, \dots, \kappa\}$ , there exists an  $(1, i, v, w)$ - or  $(2, i, v, w)$ -critical path with respect to  $\phi$ . Hence, there is an  $j$  in  $\{1, 2\}$  such that  $S_\phi(w x_j)$  are all greater than  $\tau$ , and then modify  $\phi$  by reassigning a color in  $\{\tau, \dots, \kappa\} \setminus S_\phi(w x_j)$  to  $w x_j$ , we obtain an acyclic edge coloring of  $G - vw$  and go back to Case 1. So we may assume  $x_1$  is a  $4^+$ -vertex.

Note that  $C_\phi(v_1) \subseteq \{2, \dots, \tau-1\}$  and  $C_\phi(v_2) \subseteq \{1, 3, \dots, \tau-1\}$ , it follows that  $C_\phi(v_1) \cap \{3, \dots, \tau-1\} \neq \emptyset$ , say  $3 \in C_\phi(v_1)$ . Modify  $\phi$  by reassigning 3 to  $vv_1$ , there exists an  $(3, 2)$ -bichromatic cycle containing  $vv_1$ ; otherwise, one of  $x_2$  and  $x_3$  must be a  $4^+$ -vertex, a contradiction. Hence,  $2 \in \mathcal{U}_\phi(v_1)$  and  $3 \in \mathcal{U}_\phi(v_2)$ . Hence,  $C_\phi(v_1) = \{3, \dots, \tau-1\}$  and  $C_\phi(v_2) = \{1, 4, \dots, \tau-1\}$ . Modify  $\phi$  by reassigning 4 to  $vv_1$ , we obtain an acyclic edge coloring of  $G - vw$ , we can prove that one of  $x_2$  and  $x_4$  is a  $4^+$ -vertex, a contradiction.  $\square$

**Lemma 9.** Let  $G$  be a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G) + 3$ , and let  $v$  be a 3-vertex with  $N(v) = \{w, v_1, v_2\}$ , and  $\deg(w) = \kappa - \Delta(G) + 3$ . If  $wv_1, wv_2 \in E(G)$ , then  $\deg(v_1) = \deg(v_2) = \Delta(G)$  and  $w$  is adjacent to precisely one vertex (namely  $v$ ) of degree less than  $\Delta(G) - 1$ .

**Proof.** Let  $n = \kappa - \Delta(G) + 2$ . Note that  $n \geq 5$ . Without loss of generality, assume that  $N(w) = \{v, x_1, \dots, x_n\}$ . Since  $G$  is a  $\kappa$ -deletion-minimal graph, it follows that  $G$  admits an acyclic edge coloring  $\phi$  of  $G - vw$  with  $\phi(w x_i) = i$  for  $i = 1, \dots, n$ . If  $\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v) = \emptyset$ , then  $\deg(w) + \deg(v) \leq \Delta(G) + 3 < \kappa + 2$ , which contradicts Fact 2. Hence,  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| \geq 1$ . Without loss of generality, assume that  $\phi(vv_1) = 1$ .

**Case 1.**  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| = 1$ .

Without loss of generality, we may assume that  $\phi(vv_2) = n + 1$ . For any  $i \in \{n + 2, \dots, \kappa\}$ , there exists an  $(1, i, v, w)$ -critical path with respect to  $\phi$ . Note that  $|\{n + 2, \dots, \kappa\}| = \Delta(G) - 3$ .

**Subcase 1.1.**  $\phi(wv_2) = \phi(vv_1)$ .

Thus  $S_\phi(vv_2) \supseteq \{1, n+2, \dots, \kappa\}$ . Modify  $\phi$  by erasing  $n+1$  from  $vv_2$  and assigning  $n+1$  to  $vw$ , we obtain an acyclic edge coloring  $\psi$  of  $G - vv_2$ . Note that  $|\{2, \dots, n\} \cap S_\phi(vv_2)| \leq 1$  and  $|\{2, \dots, n\} \cap S_\phi(vv_1)| \leq 2$ , it follows that  $\{2, \dots, n\} \setminus (S_\phi(vv_1) \cup S_\phi(vv_2)) \neq \emptyset$ , but every color in this set is valid for  $vv_2$  with respect to  $\psi$ , a contradiction.

**Subcase 1.2.**  $\phi(wv_2) \neq \phi(vv_1)$ .

If  $\phi(wv_2) \notin S_\phi(vv_1)$ , then modify  $\phi$  by reassigning  $\phi(wv_2)$  to  $vv_1$ , we go back to Case 1.1. So we may assume that  $\phi(wv_2) \in S_\phi(vv_1)$ . Hence,  $\deg(v_1) = \Delta(G)$  and  $S_\phi(vv_1) = \{n+2, \dots, \kappa\} \cup \{\phi(wv_1), \phi(wv_2)\}$ . Modify  $\phi$  by erasing  $n+1$  from  $vv_2$  and assigning  $n+1$  to  $vw$ , we obtain an acyclic edge coloring  $\psi$  of  $G - vv_2$ . Note that  $C_\phi(v_2) \setminus \{1, \phi(wv_1), \phi(wv_2)\} \neq \emptyset$ , but every color in  $C_\phi(v_2) \setminus \{1, \phi(wv_1), \phi(wv_2)\}$  is valid for  $vv_2$  with respect to  $\psi$ , a contradiction.

**Case 2.**  $|\mathcal{U}_\phi(w) \cap \mathcal{U}_\phi(v)| = 2$ .

Without loss of generality, we may assume that  $\phi(vv_2) = 2$ . For any  $i \in \{n+1, \dots, \kappa\}$ , there exists an  $(1, i, v, w)$ -critical path with respect to  $\phi$  or there exists an  $(2, i, v, w)$ -critical path with respect to  $\phi$ .

Let  $T_1 = \{i \mid i \in \{n+1, \dots, \kappa\} \text{ and there exists an } (1, i, v, w)\text{-critical path with respect to } \phi\}$

and  $T_2 = \{j \mid j \in \{n+1, \dots, \kappa\} \text{ and there exists an } (2, j, v, w)\text{-critical path with respect to } \phi\}$ .

Hence,  $T_1 \cup T_2 = \{n+1, \dots, \kappa\}$ .

Suppose that either  $S_\phi(vv_1) \not\supseteq \{n+1, \dots, \kappa\}$  or  $S_\phi(vv_2) \not\supseteq \{n+1, \dots, \kappa\}$ . Without loss of generality, assume that  $n+1 \notin S_\phi(vv_2)$ . By the assumption, it follows that there exists an  $(1, n+1, v, w)$ -critical path with respect to  $\phi$ . Modify  $\phi$  by reassigning  $n+1$  to  $vv_2$ , we obtain a new acyclic edge coloring  $\psi$  of  $G - vw$  with  $|\mathcal{U}_\psi(w) \cap \mathcal{U}_\psi(v)| = 1$ , and then we go back to Case 1.

So we may assume that  $S_\phi(vv_1) \supseteq \{n+1, \dots, \kappa\}$  and  $S_\phi(vv_2) \supseteq \{n+1, \dots, \kappa\}$ . In fact,  $S_\phi(vv_1) = \{n+1, \dots, \kappa\} \cup \{\phi(wv_1)\}$  and  $S_\phi(vv_2) = \{n+1, \dots, \kappa\} \cup \{\phi(wv_2)\}$ ;  $\deg(v_1) = \deg(v_2) = \Delta(G)$ .

**Subcase 2.1.**  $\phi(wv_2) \neq \phi(vv_1)$  and  $\phi(wv_1) \neq \phi(vv_2)$ .

Without loss of generality, let  $\phi(wv_1) = n-1$  and  $\phi(wv_2) = n$ . By symmetry, we may assume that  $T_1 \neq \emptyset$ . Modify  $\phi$  by reassigning 1 to  $vv_2$  and reassigning an arbitrary color  $i \in \{1, \dots, n\} \setminus \{1, n-1\}$  to  $vv_1$ , we obtain a new acyclic edge coloring  $\phi_1$  of  $G - vw$ . Note that every candidate color for  $vw$  is not valid. In other words, if assign an arbitrary color  $\theta_1$  in  $T_1$  to  $vw$ , then there exists an  $(i, \theta_1)$ -bichromatic cycle containing  $vw$ , thus  $T_1 \subseteq S_{\phi_1}(wx_i)$  for  $i = 1, \dots, n$ . Similarly, modify  $\phi$  by reassigning 2 to  $vv_1$  and reassigning an arbitrary color  $i \in \{1, \dots, n\} \setminus \{2, n\}$ , we obtain another acyclic edge coloring  $\phi_2$  of  $G - vw$ , we can prove that  $T_2 \subseteq S_{\phi_2}(wx_i)$  for  $i = 1, \dots, n$ . Now, we have  $S_{\phi_1}(wx_i) \supseteq T_1 \cup T_2 = \{n+1, \dots, \kappa\}$  for  $i = 1, \dots, n$ ; thus  $\deg(x_i) \geq \Delta(G) - 1$  for  $i = 1, \dots, n$ .

**Subcase 2.2.**  $\phi(wv_2) = \phi(vv_1)$  and  $\phi(wv_1) = \phi(vv_2)$ .

This is impossible, otherwise, there exists a bichromatic  $(1, 2)$ -cycle  $wv_2vv_1w$ .

**Subcase 2.3.**  $\phi(wv_2) = \phi(vv_1)$  and  $\phi(wv_1) \neq \phi(vv_2)$ .

Modify  $\phi$  by reassigning a color in  $\{1, \dots, n\} \setminus \{1, 2, \phi(wv_1)\}$  to  $vv_1$ , we obtain a new acyclic edge coloring of  $G - vw$ , and thus go back to Case 2.1.

**Subcase 2.4.**  $\phi(wv_2) \neq \phi(vv_1)$  and  $\phi(wv_1) = \phi(vv_2)$ .

Modify  $\phi$  by reassigning a color in  $\{1, \dots, n\} \setminus \{1, 2, \phi(wv_2)\}$  to  $vv_2$ , we obtain another acyclic edge coloring of  $G - vw$ , and thus go back to Case 2.1.  $\square$

## 4 Applications

**Theorem 4.1** (Basavaraju and Chandran [5]). If  $G$  is a graph with  $\Delta(G) \leq 4$ , then  $\chi'_a(G) \leq \Delta(G) + 2$  unless  $G$  is 4-regular.

**Proof.** Let  $G$  be a minimal counterexample to the theorem and  $\kappa = \Delta(G) + 2$ . Since the hypothesis is deletion-closed, it follows that  $G$  is a  $\kappa$ -deletion-minimal graph. Hence  $G$  is 2-connected and  $\delta(G) \geq 2$ . By Lemma 4 and Lemma 6, the graph  $G$  has no 2- and 3-vertices, hence  $G$  is 4-regular.  $\square$

**Theorem 4.2** (Basavaraju and Chandran [7]). If  $G$  is a subcubic graph, then the acyclic chromatic index of  $G$  is at most  $\Delta(G) + 1$ , unless  $G$  is 3-regular.

**Proof.** Let  $G$  be a minimal counterexample to the theorem and  $\kappa = \Delta(G) + 1$ . Since the hypothesis is deletion-closed, it follows that  $G$  is a  $\kappa$ -deletion-minimal graph. By Lemma 5, the graph  $G$  contains no 2-vertices, and then  $G$  is 3-regular.  $\square$

**Theorem 4.3.** If  $G$  is a graph with  $\text{mad}(G) < 4$ , then  $G$  has an acyclic edge coloring with  $\Delta(G) + 2$  colors.

**Proof.** Let  $G$  be a minimal counterexample to the theorem  $\kappa = \Delta(G) + 2$ . Since the hypothesis is deletion-closed, it follows that  $G$  is a  $\kappa$ -deletion-minimal graph. Hence  $G$  is 2-connected and  $\delta(G) \geq 2$ . Since  $\text{mad}(G) < 4$ , it follows that

$$\sum_{v \in V(G)} (\deg(v) - 4) < 0. \quad (1)$$

Assign the initial charge of every vertex  $v$  to be  $\deg(v) - 4$ . We design appropriate discharging rules and redistribute charges among the vertices, such that the final charge of every vertex is nonnegative, which derive a contradiction.

### The Discharging Rules:

- (R1) Every 2-vertex receives 1 from each of its  $6^+$ -neighbors;
- (R2) If  $v$  is a special 3-vertex, then  $v$  receives  $1/2$  from each of its  $5^+$ -neighbors.
- (R3) If  $v$  is a normal 3-vertex, then  $v$  receives  $1/3$  from each of its  $5^+$ -neighbors.

By Lemma 4 and Lemma 6 and the discharging rules, the final charge of every 2- or 3-vertex is zero. If  $v$  is a 4-vertex, then its final charge equals to its initial charge zero.

Let  $v$  be a 5-vertex. By Lemma 4, the vertex  $v$  is not adjacent to any 2-vertex. If  $v$  is adjacent to a special 3-vertex, then  $v$  is adjacent to at least three  $4^+$ -vertices by Lemma 6, and then its final charge is at least  $5 - 4 - 2 \times 1/2 = 0$ . If  $v$  is not adjacent to any special 3-vertex, then its final charge is at least  $5 - 4 - 3 \times 1/3 = 0$  by Lemma 8.

Let  $v$  be a  $6^+$ -vertex. If  $v$  is adjacent to at least four  $4^+$ -vertices, then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 4) = 0$ . By Lemma 3, if  $v$  is adjacent to some 2-vertices and exactly three  $4^+$ -vertices, then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 5) \times 1 - 2 \times 1/2 = 0$ . So we may assume that  $v$  is not adjacent to any 2-vertices. If  $v$  is adjacent to some special 3-vertices, then  $v$  is adjacent to at least three  $4^+$ -vertices, and then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 3) \times 1/2 = (\deg(v) - 5)/2 > 0$ . If all the  $3^-$ -vertices in  $N_G(v)$  are normal 3-vertices, then the final charge of  $v$  is at least  $\deg(v) - 4 - \deg(v) \times 1/3 = 2\deg(v)/3 - 4 \geq 0$ .  $\square$

**Theorem 4.4.** Let  $G$  be a planar graph without triangles adjacent to cycles of length 3 and 4. If every 5-cycle has at most three edges contained in triangles, then  $G$  admits an acyclic edge coloring with  $\Delta(G) + 2$  colors.

**Proof.** Let  $G$  be a minimal counterexample to the theorem and it has been embedded on the plane, let  $\kappa = \Delta(G) + 2$ . Since the hypothesis is deletion-closed, it follows that  $G$  is a  $\kappa$ -deletion-minimal graph. Hence  $G$  is 2-connected and the boundary of every face is a cycle; in other words, the boundary of a  $k$ -face is a  $k$ -cycle. By the hypothesis, we know that every 3-face is adjacent to  $5^+$ -faces.

From the Euler's formula, we have the following equality:

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -8 \quad (2)$$

Assign the initial charge of every vertex  $v$  to be  $\deg(v) - 4$  and the initial charge of every face  $f$  to be  $\deg(f) - 4$ . Clearly, the sum of the initial charges of vertices and faces is  $-8$ . We design appropriate discharging rules and redistribute charges among the vertices and faces, such that the final charge of every vertex and every face is nonnegative, which derive a contradiction.

**The Discharging Rules:**

- (R1) Every 2-vertex receives 1 from each of its  $6^+$ -neighbors;
- (R2) If  $v$  is a special 3-vertex, then  $v$  receives  $1/2$  from each of its  $5^+$ -neighbors.
- (R3) If  $v$  is a normal 3-vertex, then  $v$  receives  $1/3$  from each of its  $5^+$ -neighbors.
- (R4) If  $f$  is a 3-face, then  $f$  receives  $1/3$  from adjacent  $5^+$ -faces passing through each of its incident edges.

By Lemma 4 and Lemma 6 and the discharging rules, the final charge of every 2- or 3-vertex is zero, and the final charge of every 3-face is zero. If  $v$  is a 4-vertex, then its final charge equals to its initial charge zero.

Let  $v$  be a 5-vertex. By Lemma 4, the vertex  $v$  is not adjacent to any 2-vertex. If  $v$  is adjacent to a special 3-vertex, then  $v$  is adjacent to at least three  $4^+$ -vertices by Lemma 6, and then its final charge is at least  $5 - 4 - 2 \times 1/2 = 0$ . If  $v$  is not adjacent to any special 3-vertex, then its final charge is at least  $5 - 4 - 3 \times 1/3 = 0$  by Lemma 8.

Let  $v$  be a  $6^+$ -vertex. If  $v$  is adjacent to at least four  $4^+$ -vertices, then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 4) = 0$ . By Lemma 3, if  $v$  is adjacent to some 2-vertices and exactly three  $4^+$ -vertices, then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 5) \times 1 - 2 \times 1/2 = 0$ . So we may assume that  $v$  is not adjacent to any 2-vertices. If  $v$  is adjacent to some special 3-vertices, then  $v$  is adjacent to at least three  $4^+$ -vertices, and then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 3) \times 1/2 = (\deg(v) - 5)/2 > 0$ . If all the  $3^-$ -vertices in  $N_G(v)$  are normal 3-vertices, then the final charge of  $v$  is at least  $\deg(v) - 4 - \deg(v) \times 1/3 = 2\deg(v)/3 - 4 \geq 0$ .

If  $f$  is a 4-face, then its final charge is  $\deg(f) - 4 = 0$ . If  $f$  is a 5-face, then its final charge is at most  $5 - 4 - 3 \times 1/3 = 0$ . If  $f$  is a  $6^+$ -face, then its final charge is at least

$$\deg(f) - 4 - \deg(f) \times \frac{1}{3} = \frac{2}{3} \deg(f) - 4 \geq 0.$$

Therefore, the final charge of every vertex and every face is nonnegative, and then the sum of the final charges is nonnegative, which derive a contradiction.  $\square$

As immediate consequences of this theorem, we have the following corollaries.

**Corollary 1.** Every planar graph  $G$  without triangles adjacent to cycles of length from 3 to 5 admits an acyclic edge coloring with  $\Delta(G) + 2$  colors.

**Corollary 2 ([15]).** If  $G$  is a planar graph without 4- and 5-cycles, then  $G$  admits an acyclic edge coloring with  $\Delta(G) + 2$  colors.

**Corollary 3** ([15]). If  $G$  is a planar graph without 4- and 6-cycles, then  $G$  admits an acyclic edge coloring with  $\Delta(G) + 2$  colors.

**Corollary 4** ([12]). If  $G$  is a plane graph such that every vertex is contained in at most one 4<sup>-</sup>-face, then  $G$  admits an acyclic edge coloring with  $\Delta(G) + 2$  colors.

Recently, Wang et al. [24] proved the following result.

**Theorem 4.5.** If  $G$  is a 4-regular graph, then  $G$  admits an acyclic edge coloring with six colors.

Therefore, if  $G$  is a  $\kappa$ -deletion-minimal graph with  $\kappa \geq \Delta(G) + 2$ , then  $\Delta(G) \geq 5$ .

**Lemma 10** ([23]). Let  $G$  be a  $\kappa$ -deletion-minimal graph. If  $\kappa \geq \Delta(G) + 2$  and  $\Delta(G) \geq 5$ , then  $G$  contains no  $(4, 4, 4)$ -cycle.

**Theorem 4.6.** If  $G$  is a plane graph without intersecting triangles, then  $\chi'_a(G) \leq \Delta(G) + 3$ .

**Proof.** Let  $G$  be a minimal counterexample to the theorem and  $\kappa = \Delta(G) + 2$ . Since the hypothesis is deletion-closed, it follows that  $G$  is a  $\kappa$ -deletion-minimal graph. Hence  $G$  is 2-connected and the boundary of every face is a cycle; in other words, the boundary of a  $k$ -face is a  $k$ -cycle.

From the Euler's formula, we have the following equality:

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -8 \quad (3)$$

Assign the initial charge of every vertex  $v$  to be  $\deg(v) - 4$  and the initial charge of every face  $f$  to be  $\deg(f) - 4$ . Clearly, the sum of the initial charges of vertices and faces is  $-8$ . We design appropriate discharging rules and redistribute charges among the vertices and faces, such that the final charge of every vertex and every face is nonnegative, which derive a contradiction.

**The Discharging Rules:**

- (R1) Every 2-vertex receives 1 from each of its 7<sup>+</sup>-neighbors;
- (R2) If  $v$  is a special 3-vertex, then  $v$  receives 1/2 from each of its 6<sup>+</sup>-neighbors.
- (R3) If  $v$  is a normal 3-vertex, then  $v$  receives 1/3 from each of its 6<sup>+</sup>-neighbors.
- (R4) Let  $f$  be a 3-face in  $G$ . If  $f$  is incident with exactly one 5<sup>+</sup>-vertex, then  $f$  receives 1 from this 5<sup>+</sup>-vertex; if  $f$  is incident with at least two 5<sup>+</sup>-vertices, then  $f$  receives 1/2 from each of its incident 5<sup>+</sup>-vertex.

By Lemma 4 and Lemma 6 and the discharging rules, the final charge of every 2- or 3-vertex is zero. If  $v$  is a 4-vertex, then its final charge equals to its initial charge zero.

Let  $v$  be a 5-vertex. By Lemma 4, the vertex  $v$  is not adjacent to any 2-vertex. By the discharging rules, the vertex  $v$  can only send charges to its incident 3-face, and thus the final charge of  $v$  is at least  $5 - 4 - 1 = 0$ .

Let  $v$  be a 6-vertex. By Lemma 4, the vertex  $v$  is not adjacent to any 2-vertices. By Lemma 8 and the discharging rules, the vertex  $v$  sends charges to at most four 3-vertices or one 3-face, the final charge of  $v$  is at least  $6 - 4 \times 1/2 - 1 = 0$ .

Let  $v$  be a 7<sup>+</sup>-vertex. If  $v$  is adjacent to at least five 4<sup>+</sup>-vertices, then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 5) - 1 = 0$ . By Lemma 3, if  $v$  is adjacent to some 2-vertices and exactly four 4<sup>+</sup>-vertices, then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 6) \times 1 - 2 \times 1/2 - 1 = 0$ . So we may assume that  $v$  is not adjacent to any 2-vertices. If  $v$  is adjacent to some special 3-vertices, then  $v$  is adjacent to at least four 5<sup>+</sup>-vertices, and then its final charge is at least  $\deg(v) - 4 - (\deg(v) - 4) \times 1/2 - 1 =$



$(\deg(v) - 6)/2 > 0$ . Thus we assume all the  $3^-$ -vertices in  $N_G(v)$  are normal 3-vertices. If  $v$  is incident with a 3-face, then it is adjacent to at least one  $4^+$ -vertices, and then the final charge of  $v$  is at least  $\deg(v) - 4 - (\deg(v) - 1) \times 1/3 - 1 = (2\deg(v) - 14)/3 \geq 0$ . If  $v$  is not incident with any 3-face, then the final charge of  $v$  is at least  $\deg(v) - 4 - \deg(v)/3 = (2\deg(v) - 12)/3 > 0$ .

If  $f$  is a  $4^+$ -face, then its final charge is  $\deg(f) - 4 \geq 0$ . By Lemma 4 and Lemma 7 and Lemma 10, every 3-face is incident with at least one  $5^+$ -vertex, then its final charge of every 3-face is nonnegative.

Therefore, the final charge of every vertex and every face is nonnegative, and then the sum of the final charges is nonnegative, which derive a contradiction.  $\square$

## 5 Concluding remarks

Lemma 3 does not provide any local structures on the  $\kappa$ -deletion-minimal graph  $G$  if  $\kappa = \Delta(G)$ . Here, we consider one extremal case:

**Theorem 5.1.** Let  $G$  be a graph with  $\Delta(G) \geq 3$ . If all the  $3^+$ -vertices are independent, then  $\chi'_a(G) = \Delta(G)$ .

**Proof.** Let  $G$  be a minimal counterexample to the theorem. Suppose that  $G$  is not 2-connected, each block is a smaller graph with maximum degree at least three or a graph with maximum degree at most two, so each block admits an acyclic edge coloring with  $\Delta(G)$  colors, and thus  $G$  also admits an acyclic edge coloring with  $\Delta(G)$  colors. Hence  $G$  is 2-connected and the minimum degree is at least two.

**Claim 1.** Every  $3^+$ -vertex is a vertex with maximum degree.

**Proof.** Suppose that  $v$  is a vertex with  $3 \leq \deg(v) < \Delta(G)$  and  $N_G(v) = \{v_0, v_1, \dots, v_n\}$ . We may assume that the graph  $G - vv_0$  admits an acyclic edge coloring  $\phi$  with  $\Delta(G)$  colors and  $\phi(vv_i) = i$  for  $i = 1, \dots, n$ . Note that  $|\mathcal{U}_\phi(v) \cup \mathcal{U}_\phi(v_0)| < \Delta(G)$  and  $C_\phi(vv_0) \neq \emptyset$ , every color in  $C_\phi(vv_0)$  is not valid with respect to  $\phi$ . Thus we may assume that  $\phi(v_0w_0) = 1$  and  $|C_\phi(vv_0)| \geq 2$ . There exists an  $(1, \theta, v, v_0)$ -critical path for every color  $\theta$  in  $C_\phi(vv_0)$ , and thus  $C_\phi(vv_0) \subseteq S_\phi(vv_1)$ , which contradicts with the fact that  $v_1$  is a 2-vertex.  $\square$

Suppose that there exists an edge  $xy$  with  $\deg(x) = \deg(y) = 2$ . By the 2-connectedness of  $G$ , the edge  $xy$  is not connected in any triangles, thus  $G/xy$  is a simple graph and all the  $3^+$ -vertices are also independent, so  $G/xy$  admits an acyclic edge coloring with  $\Delta(G)$  colors. It is easy to see that this coloring can be extended to an acyclic edge coloring of  $G$  with  $\Delta(G)$  colors, a contradiction. Now we know that the graph  $G$  is bipartite. Let  $G$  be the bipartite graph with bipartition  $X$  and  $Y$ , where  $X$  are the collection of  $\Delta(G)$ -vertices and  $Y$  are the 2-vertices.

**Claim 2.** For any acyclic edge coloring  $\phi$  of  $G - xy_0$ , we have  $\mathcal{U}_\phi(x) \cap \mathcal{U}_\phi(y_0) = \emptyset$ .

**Proof.** Let  $N_G(x) = \{y_0, y_1, \dots, y_n\}$  and  $\phi(xy_i) = i$  for  $i = 1, \dots, n$ . Without loss of generality, we may assume that  $\phi(xy_1) = \phi(y_0w_0) = 1$ . The only candidate color  $\theta$  for  $xy_0$  is not valid, there exists an  $(1, \theta, x, y_0)$ -critical path with respect to  $\phi$ . If there exists a vertex  $y_i$  with  $i \geq 2$  such that  $\theta \in \mathcal{U}_\phi(y_i)$ , then modify  $\phi$  by exchanging the colors on  $xy_1$  and  $xy_i$ , we obtain a new acyclic edge coloring of  $G - xy_0$ , but now  $\theta$  is valid for  $xy_0$ , a contradiction. Hence,  $\theta \in C_\phi(y_i)$  for  $i = 2, \dots, n$ . Let  $\phi(xy_2) = \beta$  and  $\phi(y_2w_2) = \alpha$ . Modify  $\phi$  by reassigning  $\theta$  to  $xy_2$ , we obtain a proper edge coloring  $\phi^*$  of  $G - xy_0$ . In fact, the resulting coloring  $\phi^*$  is an acyclic edge coloring, otherwise, there exists an  $(\theta, \alpha)$ -bichromatic cycle. This bichromatic cycle is an  $(\theta, 1)$ -bichromatic cycle  $xy_2w_2y_1x$  with respect to  $\phi^*$ , and thus there exists an  $(1, \theta, x, y_2)$ -critical path, which contradicts with Fact 1. But now the color  $\beta$  is valid for  $xy_0$  with respect to  $\phi^*$ .  $\square$

Let  $x_0y_0$  be an edge of  $G$  with  $x_0 \in X$  and  $y_0 \in Y$ . The graph  $G - x_0y_0$  admits an acyclic edge coloring  $\phi$  with  $\Delta(G)$  colors. By Claim 2, we have  $\mathcal{U}_\phi(x_0) \cap \mathcal{U}_\phi(y_0) = \emptyset$ . We may assume that  $\mathcal{U}_\phi(x_0) = \{2, \dots, \Delta(G)\}$  and  $\mathcal{U}_\phi(y_0) = \{1\}$ . Recall that  $\mathcal{U}_\phi(x) = \{1, \dots, \Delta(G)\}$  for every vertex  $x$  in  $X$ . Let  $y_0x_1y_1 \dots$  is the

maximal  $(1, 2)$ -path with respect to  $\phi$ . This path ends with an edge  $x_k y_k$  which is colored with 2, since all the vertices in  $X$  seeing the color 2. Modify  $\phi$  by assigning 1 to  $x_i y_i$  and assigning 2 to  $x_i y_{i+1}$ , where  $i = 0, \dots, k-1$ , we obtain an acyclic edge coloring  $\psi$  of  $G - x_k y_k$ . But  $\mathcal{U}_\psi(x_k) \cap \mathcal{U}_\psi(y_k) \neq \emptyset$ , which contradicts with [Claim 2](#).  $\square$

A  $\kappa$ -minimal graph  $G$  is one with maximum degree at most  $\kappa$ , and it has no acyclic edge coloring with  $\kappa$  colors, but every graph  $H$  with  $\Delta(H) \leq \Delta(G)$  and  $|V(H)| + |E(H)| < |V(G)| + |E(G)|$  does has an acyclic edge coloring with  $\kappa$  colors. Obviously, every proper subgraph of a  $\kappa$ -minimal graph admits an acyclic edge coloring with  $\kappa$  colors. In other words, every  $\kappa$ -minimal graph is also a  $\kappa$ -deletion-minimal graph.

**Lemma 11.** Let  $G$  be a  $\kappa$ -minimal graph with  $\kappa \geq \Delta(G) + 1$ . If  $v_0$  is a 2-vertex of  $G$ , then  $v_0$  is contained in a triangle.

**Proof.** Let  $N_G(v_0) = \{v, w\}$  and  $e = vv_0$ . By contradiction, suppose that  $v$  and  $w$  are nonadjacent. The graph  $G/e$  is a simple graph, it admits an acyclic edge coloring  $\phi$  with  $\kappa$  colors. Note that  $\Delta(G/e) \leq \Delta(G)$ , there is a color  $\theta$  in  $C_\phi(v)$ . Extend  $\phi$  by assigning  $\theta$  to  $vv_0$ , we obtain an acyclic edge coloring of  $G$ , a contradiction.  $\square$

**Lemma 12.** Let  $G$  be a  $\kappa$ -minimal graph with  $\kappa \geq \Delta(G) + 2$ . If  $v$  is a 3-vertex in  $G$ , then every neighbor of  $v$  is a  $(k - \Delta(G) + 3)^+$ -vertex.

**Proof.** By [Lemma 7](#), every neighbor of  $v$  is a  $(k - \Delta(G) + 2)^+$ -vertex. By [Lemma 6](#), if  $v$  is adjacent to a  $(k - \Delta(G) + 2)$ -vertex  $w$ , then the edge  $vw$  is not contained in any triangles of  $G$ , and thus the graph  $G/vw$  is a simple graph with maximum degree  $\Delta(G)$  since the degree of the new vertex has degree  $k - \Delta(G) + 3$ . Thus  $G/vw$  admits an acyclic edge coloring  $\phi$  with  $\kappa$  colors, and this coloring can be easily extended to an acyclic edge coloring of  $G$ , a contradiction.  $\square$

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